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Periodic solution of a single system of differential equations in partial derivatives

Altinshash Bekbauova *, Aksaule Baibaktina, Bibigul Omarova, Bayan Abilmazhinova, Lida Sultangaliyeva, Gulzhan Erzhanova, Madina Tleubergenova

K. Zhubanov Aktobe Regional State University, Aktobe, Kazakhstan

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ABSTRACT

The main difficulty of the Cauchy equation is that there is no domain in which the desired solution must be determined. This situation leads to the complexity of finding the answer. The study presents a solution of the Cauchy problem at any values. The achievement of the set goal will enable solving one of the key problems of gas dynamics. To that end, it is necessary to define the solution, since a solution in a classical form is nonexistent for this type of equations. We presented an algorithm for the construction of periodic, in terms of variables, solutions of the system in first order partial derivatives. The study found a sufficient condition of existence of periodic, in terms of variables, solutions in the broad sense of differential equation systems in partial derivatives.

1. Introduction

Differential equations are a basic mathematical tool that is used to model various physical laws and phenomena (Perko, 2013; Filippov, 2013; Browder, 2016). Systems of linear and quasilinear hyperbolic equations often emerge during the mathematical modeling of various oscillatory processes (Kang, 2014) that occur in a solid medium, when studying the flow of combustible gases and liquids, filtration problems, and the shallow water theory. Triangular block linear systems in first order partial derivatives simplify the construction of principal matrix solutions of linear systems, since the first equation of the system can be regarded as an equation with an identical main part.

Nowadays, the theory of differential equations is the basis of engineering, physical, and chemical calculations in science and industry (Butcher, 2016; Polyanin and Nazaikinskii, 2015; Edwards and Penney, 2014). Thus, the theory of differential equations serves as a source for modern exact sciences (Nemytskii, 2015; Bluman and Kumei, 2013). Differential equations also play a major role in other sciences, such as economics, biology, electrical engineering, etc. In fact, they are encountered everywhere, where a quantitative description of phenomena is required (Amel'kin, 1990). The classical solutions of nonlinear equations are characterized by an infinite increase of the value of derivatives, which is called a gradient catastrophe. The essence of this property lies in the fact that with any arbitrarily smooth initial values, the first derivatives remain limited for a finite time only. At a certain $t_0 > 0$, they become unlimited. At $t > t_0$, a classical solution of the set Cauchy problem is nonexistent (for instance, a shockwave that is formed by a contraction wave). Thus, in order to determine the solution of the Cauchy problem at any t values, i.e., in general (the main problem of gas dynamics), it is necessary to define the solution, since, as was mentioned above, a solution in its classical form is nonexistent.

2. Materials and methods

In most physical problems, the definition of the generalized solutions is dictated by the setting of the problem (for instance, in gas dynamics, the main physical laws are the laws of conservation of mass, impulse, and energy, while the generalized solution is defined as a flow that satisfies these main laws). Based on work by Friedrichs (1948), the generalized solution of a system in partial derivatives is called a broad-sense solution. Consider a hyperbolic linear system

$$D_1 x = P_{11}(t,\varphi,\psi)x + f_1(t,\varphi,\psi) \qquad (1)$$
$$D_2 x = P_{21}(t,\varphi,\psi)x + P_{22}(t,\varphi,\psi)y + f_2(t,\varphi,\psi)$$

* Corresponding Author.
Email Address: mirra478@mail.ru (A. Bekbauova)
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where

$t \in (-\infty, +\infty)R, \varphi = (\varphi_1, \dots, \varphi_m) \in R^m,$
$\psi = (\psi_1, \dots, \psi_k) \in R^k$
$x = (x_1, \dots, x_{n1}), y = (y_1, \dots, y_{n2})$

are the sought vector functions, D_1 and D_2 are differential operators

$$D_1 = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_{1j}(t,\varphi,\psi)\frac{\partial}{\partial\varphi j} + \sum_{j=1}^{k} b_{1j}(t,\varphi,\psi)\frac{\partial}{\partial\psi j},$$
$$D_2 = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_{2j}(t,\varphi,\psi)\frac{\partial}{\partial\varphi j} + \sum_{j=1}^{k} b_{2j}(t,\varphi,\psi)\frac{\partial}{\partial\psi j},$$

$P_{ij}(t,\varphi,\psi) - n_i \times n_j$ – matrices, $(i,j = 1{,}2)$, which are periodic and limited

$$\|P_{ij}\| \le k_{ij}, \quad (2)$$
$$P_{ij}(t+\theta, \varphi + q\omega, \psi) =$$
$$= P_{ij}(t,\varphi,\psi) \in C(R \times R^m \times R^k)$$
$$\|f_i\| \le k_{ij},$$
$$f_1(t+\theta, \phi + q\omega, \psi =$$
$$f_1(t,\varphi,\psi) \in C(R \times R^m \times R^{k'} \quad (3)$$
$$a_{11}(t,\varphi,\psi), a_{21}(t,\varphi,\psi),$$
$$b_{11}(t,\varphi,\psi), b_{21}(t,\varphi,\psi) -$$

continuous vector dimension functions, m, k, respectively, which are periodic and smooth

$$q_1(t+\theta, \phi + q\omega\psi) =$$
$$q_1(t,\varphi,\psi) \in C_{t,\phi,\psi}^{(0,11)}(R \times R^m \times R^{k'}$$
$$B_{1l}(t+\theta, \phi + q\omega, \psi) =$$
$$b_{1l}(t,\varphi,\psi) \in C_{t,\phi,\psi}^{(0,11)}(R \times R^m \times R^{k'} \quad (4)$$

and limited, with a norm that maximizes the Euclidian metrics of the vector function

$$\|\alpha_{ij}\| \le \alpha_{00}, \left\|\frac{\partial}{\partial\varphi}\alpha_{ij}\right\| \le \alpha_{10}, \left\|\frac{\partial}{\partial\psi}\alpha_{ij}\right\| \le \alpha_{01},$$
$$\|b_{ij}\| \le \beta_{00},$$
$$\left\|\frac{\partial}{\partial\varphi}b_{ij}\right\| \le \beta_{10}, \left\|\frac{\partial}{\partial\varphi}b_{ij}\right\| \le \beta_{01}$$

for all integer vectors $q = (q_1, \dots, q_m) \in Z \times \dots \times Z = Z^m$, Z- is a set of integers. Periods $\omega_0 = \theta, \omega_1, \dots, \omega_m$ are rationally incommensurable and constant, $q\omega = (q_1\omega_1, q_2\omega_2, \dots, q_m\omega_m)$ is the vector of multiple periods, $\omega = (\omega_1, \dots, \omega_m)$, $i,j,l = 1{,}2$. $\alpha_{00}, \beta_{00}, \alpha_{10}, \beta_{10}, \alpha_{01}, \beta_{01}$ are positive constants.

3. Definition

A continuous in $R \times R^m \times R^k$ function $z(t,\varphi,\psi) = (x(t,\varphi,\psi), y(t,\varphi,\psi))$ is called a multiperiodic, in terms of variables, solution of system (1) in the broad sense if it is multiperiodic at t, φ with a period vector of (θ,ω) limited in all variables and continuously differentiated for variable t along the characteristics
$\{\lambda^1(t,t_0,\varphi_0,\psi_0), \xi^1(t,t_0,\varphi_0,\psi_0)\}$

and

$\{\lambda^2(t,t_0,\varphi_0,\psi_0), \xi^2(t,t_0,\varphi_0,\psi_0)\}$;
at that, for full derivatives for t, the following identities are satisfied

$$\begin{cases} \frac{dx}{dt} P_{11}x \\ \frac{dy}{dt} = P_{21}x + P_{22}y \end{cases} \quad (5)$$

where

$x = x(t, \lambda^1(t,t_0,\varphi_0,\psi_0), \xi^1(t,t_0,\varphi_0,\psi_0))$
$P_{11} = P_{11}(t, \lambda^1(t,t_0,\varphi_0,\psi_0), \xi^1(t,t_0,\varphi_0,\psi_0))$
$y = y(t, \lambda^2(t,t_0,\varphi_0,\psi_0), \xi^2(t,t_0,\varphi_0,\psi_0))P_{22} =$
$P_{22}(t, \lambda^2(t,t_0,\varphi_0,\psi_0), \xi^2(t,t_0,\varphi_0,\psi_0))$
$P_{21} = P_{21}(t, \lambda^1(t,t_0,\varphi_0,\psi_0), \xi^1(t,t_0,\varphi_0,\psi_0))$
$P_{21} = P_{21}(t, \lambda^2(t,t_0,\varphi_0,\psi_0), \xi^2(t,t_0,\varphi_0,\psi_0))$

The task is to study the existence of multiperiodic, in terms of variables, broad-sense solutions of system (1).

In order to find the characteristics of the function of differential operators, the following systems of ordinary differential equations:

$$\begin{cases} \frac{d\varphi}{dt} = a_{11}(t,\varphi,\psi) \\ \frac{d\psi}{dt} = b_{11}(t,\varphi,\psi) \end{cases} \qquad \begin{cases} \frac{d\varphi}{dt} = a_{21}(t,\varphi,\psi) \\ \frac{d\psi}{dt} = b_{21}(t,\varphi,\psi) \end{cases}$$

solutions

$$\varphi = \lambda^1(t,t_0,\varphi_0,\psi_0), \psi = \xi^1(t,t_0,\varphi_0,\psi_0)$$

and

$$\varphi = \lambda^2(t,t_0,\varphi_0,\psi_0),$$
$$\psi = \xi^2(t,t_0,\varphi_0,\psi_0)$$

of the two systems of ordinary differential equations with initial data $(t,t_0,\varphi_0,\psi_0) \in R \times R^m \times R^k$ under conditions (4) are determined globally at $t \in R$.

Let us find the principal matrix solution of a homogenous system that corresponds with system (1) in a fashion similar to (Bekbauova et al., 2010). To that end, consider the following integral equations:

$$X(t,\phi,\psi,t_0,\lambda^1(t_0,t,\varphi,\psi),\zeta^1(t_0,t,\varphi,\psi)) =$$
$$X_0(t,\phi,\psi,t_0,\lambda^1(t_0,t,\varphi,\psi),\zeta^1(t_0,t,\varphi,\psi)) +$$
$$+ \int_{t_0}^{t} X_0(t,\phi,\psi,s,\lambda^1(s,t,\phi,\psi),\zeta^1(s,t,\phi,\psi)).$$
$$P_{12}(s,t,\phi,\psi),\zeta^1(s,t,\phi,\psi)) \times$$
$$\times Y\begin{pmatrix} s,\lambda^1(s,t,\phi,\psi),\zeta^1(s,t,\phi,\psi),t_0, \\ \lambda^1(t_0t,\phi,\psi),\zeta^1(t_0t,\phi,\psi) \end{pmatrix} ds,$$
$$Y(t,\phi,\psi,t_0\lambda^2(t_0t,\phi,\psi),\zeta^2(t_0t,\phi,\psi))$$
$$= Y_0(t,\phi,\psi,t_0\lambda^2(t_0t,\phi,\psi),\zeta^2(t_0t,\phi,\psi))$$
$$+ \int_{t_0}^{t} Y_0(t,\phi,\psi,s,\lambda^2(s,t,\phi,\psi),\zeta^2(s,t,\phi,\psi))$$
$$P_{21}(s,\lambda^2(s,t,\phi,\psi),\zeta^2(s,t,\phi,\psi)) \times$$
$$\times X\begin{pmatrix} s,\lambda^2(s,t,\phi,\psi),\zeta^2(s,t,\phi,\psi),t_0, \\ \lambda^2(t_0t,\phi,\psi),\zeta^2(t_0t,\phi,\psi) \end{pmatrix} ds,$$

4. Results and discussions

In order to solve the system of integral equations, we use the method of successive approximation and study the convergence of series, similar to (Bekbauova et al., 2010). Consequently, the diagonal matrix

$$Z^1\begin{pmatrix} t,\phi,\psi,t_0\lambda^1(t_0t,\phi,\psi),\zeta^1(t_0t,\phi,\psi) \\ \lambda^2(t_0t,\phi,\psi),\zeta^2(t_0t,\phi,\psi) \end{pmatrix} = diag[X,Y],$$

is a broad-sense solution of the corresponding linear homogenous system (1) and satisfies the initial condition

$$Z^1(t_0, \varphi, \psi, t_0, \varphi, \psi, \varphi, \psi)) = E$$

which we will call the principal matrix solution of system (1) in the broad sense, where E is an identity matrix.

Assume that the principal matrix solution has the following property

$$\left|Z^1\begin{pmatrix} t,\phi,\psi,t_0\lambda^1(t_0t,\phi,\psi),\zeta^1(t_0t,\phi,\psi), \\ \lambda^2(t_0t,\phi,\psi),\zeta^2(t_0t,\phi,\psi) \end{pmatrix}\right| \leq B_1 e^{-\gamma(t-t_0)} \qquad (5)$$

where $t \geq t_0, B = const \geq 1,\ \ \gamma = \min\{\gamma_1, \gamma_2\} > 0$.

Since under conditions (2), (4), and (5), any broad-sense solution $z(t, \varphi, \psi) = (x(t, \varphi, \psi), y(t, \varphi, \psi))$ of a linear homogenous system can be presented in the following form

$$z(t, \varphi, \psi) =$$
$$\left.\begin{matrix} Z^1(t,\varphi,\psi,t_0,\lambda^1(t_0,t,\varphi,\psi),\zeta^1(t_0,t,\varphi,\psi), \\ \lambda^2(t_0,t,\varphi,\psi),\zeta^2(t_0,t,\varphi,\psi) \end{matrix}\right) \times$$
$$\left.\begin{matrix} \times u(\lambda^1(t_0,t,\varphi,\psi),\zeta^1(t_0t,\phi,\psi), \\ \lambda^2(t_0,t,\varphi,\psi),\zeta^2(t_0,t,\varphi,\psi) \end{matrix}\right)$$

where

$$\left.\begin{matrix} (\lambda^1(t_0,t,\varphi,\psi),\zeta^1(t_0t,\phi,\psi), \\ (t_0,t,\varphi,\psi),\zeta^2(t_0,t,\varphi,\psi) \end{matrix}\right) =$$
$$= \begin{pmatrix} v(\lambda^1(t_0,t,\varphi,\psi),\zeta^1(t_0t,\phi,\psi)) \\ w(\lambda^2(t_0,t,\varphi,\psi),\zeta^2(t_0t,\phi,\psi)) \end{pmatrix}$$

The initial function satisfies the following condition:

$$u(\varphi, \psi) = \big(u_1(\varphi, \psi), \ldots, u_n(\varphi, \psi)\big),$$
$$u(\varphi, +q\omega, \psi) = u(\varphi, \psi) \in C(R^m \times R^k), q \in Z^m,$$

with a norm of

$$\|u\| = \sup_{R^m \times R^k} \sqrt{\textstyle\sum_{j-1}^{n} u_j^2(\varphi, \psi),}$$

Theorem 1: Under conditions (2), (4), and (5), the linear system has no (θ, ω) periodic solutions except the zero-state solution.

Theorem 2: Assume conditions (2), (3), (4), and (5) are met. Then the linear nonhomogeneous system (1) has a single multiperiodic, in terms of variables, broad-sense solution.

$$x^*(t, \varphi, \psi) = \int_{-\infty}^{t} X$$
$$(t, \varphi, \psi, s, \lambda^1(s, t, \varphi, \psi), \xi^1(s, t, \varphi, \psi)$$
$$\times f_1\big(s, \lambda^1(s, t, \varphi, \psi), \xi^1(s, t, \varphi, \psi)\big)ds$$
$$\times f_1\big(s, \lambda^1(s, t, \varphi, \psi), \xi^1(s, t, \varphi, \psi)\big)ds$$
$$(t, \varphi, \psi, s, \lambda^2(s, t, \varphi, \psi), \xi^2(s, t, \varphi, \psi))$$
$$\times f_2\big(s, \lambda^2(s, t, \varphi, \psi), \xi^2(s, t, \varphi, \psi)\big)ds$$
$$\|x^*(t, \varphi, \psi)\| \leq \frac{BK_1}{\gamma}$$

$$\|y^*(t, \varphi, \psi)\| \leq \frac{BK_2}{\gamma}$$

where

$$\|f_1\| = \sup_{R \times R^m \times R^k} \|f_1(t, \varphi, \psi)\| = K_1$$
$$\|f_2\| = \sup_{R \times R^m \times R^k} \|f_2(t, \varphi, \psi)\| = K_2$$

5. Conclusion

Thus, the proposed estimations reduce the number of calculations in comparison to other methods. The achievement of the set goal enables theoretically solving one of the most complex problems of gas dynamics.

The study showed the conditions of existence and uniqueness of a multiperiodic, in terms of variables, broad-sense solution of linear systems of differential equations in first order partial derivatives with an identical main part.

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